Optimization for GDR IFM

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Scientific day on optimization

GDR IFM

Informatique Fondamentale et ses Mathématiques (IFM)

Direction: **Pierre Fraigniaud and Guillaume Theyssier**

Problems and motivations: **Foundations of computer science**

Strong interaction with mathematics

Scientific Domains

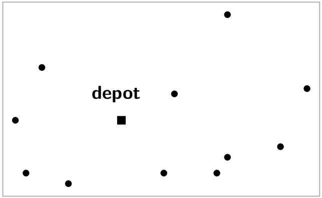
- **Combinatorics, graphs, and dynamic systems**
  ALEA, COMBALG, Graphes, SDA2, SEQBIM
- **Computability, complexity, algorithmic, and quantum computation**
  COA, Calculabilités, IQ
- **Computer algebra, arithmetic, and cryptography**
  ARITH, CF, C2
- **Programs, verification, proof, automaton, and logic**
  DAAL, VERIF, SCALP, LHC, BIOSS
- **Geometry and image**
  GDMM, MG, GEOALGO

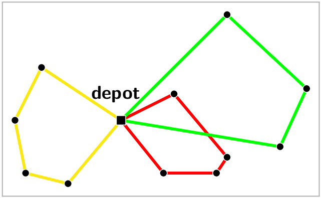
Capacitated Vehicle Routing Problem (CVRP)

Input:

- **depot**
- **set of terminals**
- **capacity k**

Minimize total length of tours





Two models: unit demand v.s. arbitrary unsplittable demand

Fundamental problem in operations research

Unit Demand

- 1985 Haimovich and Rinnooy Kan
- 1990 Altinkemer and Gavish
- 1997 Asano, Katoh, Tamaki, and Tokuyama
- 1998 Hamaguchi and Katoh
- 2001 Asano, Katoh, and Kawashima
- 2006 Bompadre, Dror, and Orlin
- 2010 Adamaszek, Czumaj, and Lingas
- 2010 Das and Mathieu
- 2017 Becker, Klein, and Saulpic
- 2018 Becker, Klein, and Saulpic
- 2018 Becker
- 2019 Becker, Klein, and Schild
- 2019 Becker and Paul
- 2020 Cohen-Addad, Filtser, Klein, and Le
- 2022 Blauth, Traub, and Vygen
- 2022 Jayaprakash and Salavatipour
- 2022 Jayaprakash and Salavatipour
- 2022 Mathieu and Z.
- 2023 Mömke and Z.
- 2023 Mathieu and Z.
- 2023 Dufay, Mathieu, and Z.
- 2024 Nie and Z.

Unsplittable Demand

- 1981 Golden and Wong
- 1987 Altinkemer and Gavish
- 1991 Labbé, Laporte, and Mercure
- 2021 Blauth, Traub, and Vygen
- 2022 Friggstad, Mousavi, Rahgoshay, and Salavatipour
- 2023 Grandoni, Mathieu, and Z.
- 2023 Mathieu and Z.

general metrics

Euclidean plane

trees

planar graphs

graphs of bounded treewidth

graphs of bounded highway dimension

graphic metrics

Outline

Unit-demand CVRP on trees

- $(1+\epsilon)$-**approximation** [Mathieu and Z. 2022]

Unit-demand CVRP in the Euclidean plane – random setting

- **1.55-approximation** [Nie and Z. 2024]

Unit-demand CVRP in graphic metrics

- **1.95-approximation** [Mömke and Z. 2023]

Unsplittable CVRP on trees

- $(1.5+\epsilon)$-**approximation** [Mathieu and Z. 2023]
- 1.5-hardness [Golden and Wang 1981]

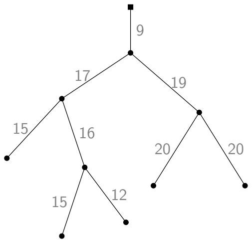
Unsplittable CVRP in the Euclidean plane

- $(2+\epsilon)$-**approximation** [Grandoni, Mathieu, and Z. 2023]

Part I:
Unit demand CVRP on trees

Unit demand CVRP on trees

- 1.5-approximation [HK'98]
- 1.351-approximation [AKK'01]
- 1.333-approximation [Bec'18]
- bicriteria $(1+\epsilon)$-approximation [BP'19]
- quasi-polynomial time $(1+\epsilon)$-approximation [JS'22]



Mathieu and Z. 2022

polynomial time $(1+\epsilon)$-approximation (PTAS)

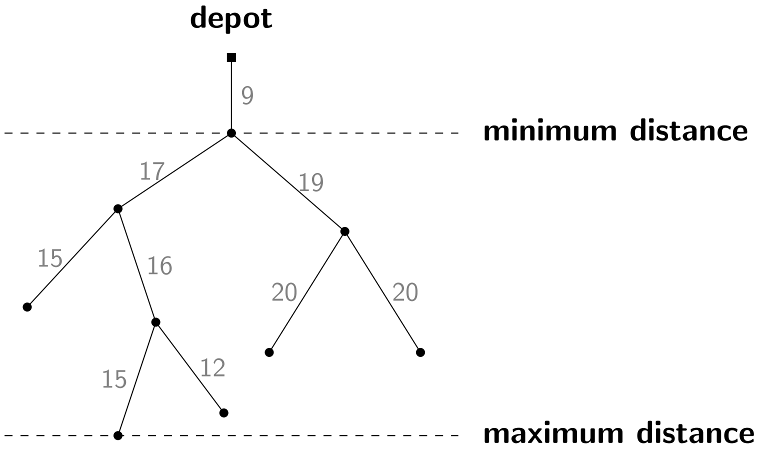
First PTAS for CVRP in any non-trivial metric

Unit demand CVRP on trees

Jayaprakash and Salavatipour [SODA 2022]:

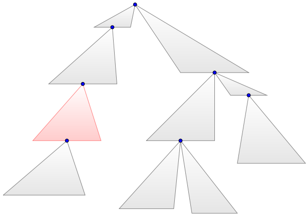
"*it is not clear if it (their algorithm) can be turned into polynomial time without significant new ideas.*"

Preprocessing: bounded distance property



Can assume: $\dfrac{\textbf{minimum distance}}{\textbf{maximum distance}} > \epsilon$

Decomposing the tree into components



Each **component** has $\approx k/\epsilon$ terminals.

Structure Theorem

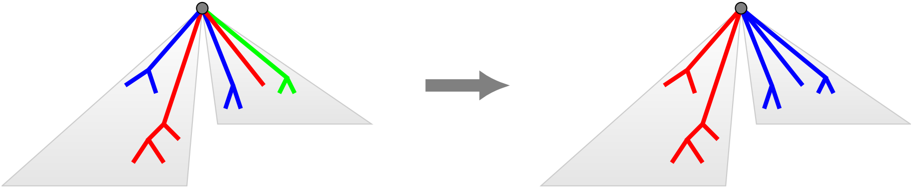
There is a near-optimal solution such that each subtour in a component visits at least $\epsilon \cdot k$ terminals.

Proof of the Structure Theorem (1/2)

Definition: a subtour is **small** if it visits less than $\epsilon \cdot k$ terminals.

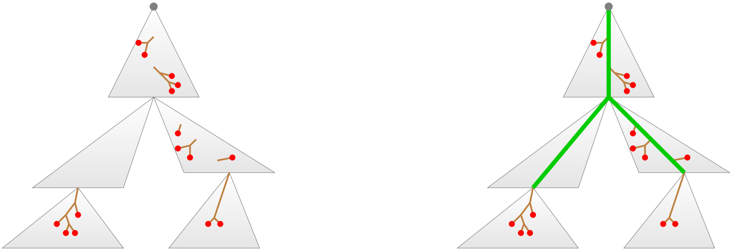
How to eliminate small subtours?

1. Detach small subtours
2. Combine small subtours within components
3. Reassign combined subtours



After reassignment, some tours may exceed capacity.

Proof of the Structure Theorem (2/2)



4. Remove some subtours
5. Include spines of the components

} traveling salesman tour

6. **Partition the traveling salesman tour into smaller tours**

Structure Theorem

There is a near-optimal solution such that each subtour in a component visits at least $\epsilon \cdot k$ terminals.

Towards a dynamic program

Computing solutions inside each component: **polynomial time**

Combining solutions from different components: **exponential time**

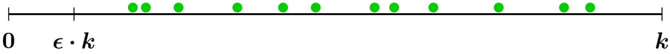
Q: How to improve the running time?

A: Adaptive rounding to reduce the number of subtour demands.

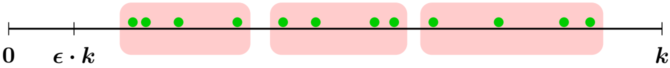
Adaptive rounding

At each vertex:

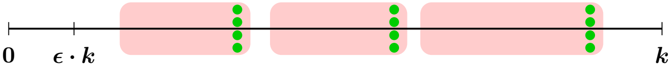
1. **Sort** subtour demands



2. **Make groups** of equal cardinality



3. **Round up** to maximum demand in group



Analysis on the adaptive rounding

Theorem [Jayaprakash and Salavatipour]

There is a near-optimal solution in which the subtour demands can be rounded up to **polylogarithmic** many values.

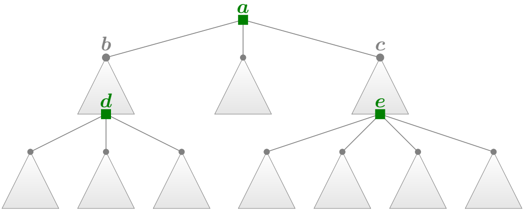
$\Longrightarrow$ **quasi-polynomial time**

Theorem [Mathieu and Z.]

There is a near-optimal solution in which the subtour demands can be rounded up to **constant** many values.

$\Longrightarrow$ **polynomial time**

Dynamic program



Order of computation:

1. each component
2. subtrees rooted at d and e by **adaptive rounding**
3. subtrees rooted at b and c
4. subtree rooted at a by **adaptive rounding**

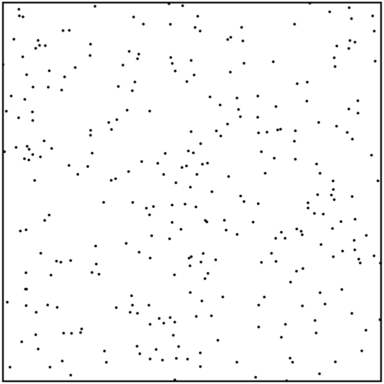
Mathieu and Z. 2022

polynomial time $(1 + \epsilon)$-approximation for unit demand tree CVRP

Part II:
Unit-demand CVRP in the Euclidean plane

Random Setting in the Euclidean Plane

Terminals: **independent, identically distributed** random points in $[0,1]^2$



CVRP in the Random Setting

Haimovich and Rinnooy Kan 1985

2-approximation

Bompadre, Dror, and Orlin 2007

1.995-approximation

Mathieu and Z. 2023

1.915-approximation

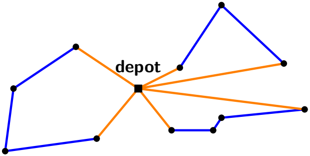
Nie and Z. 2024

1.55-approximation

Analysis: Warm Up [Haimovich and Rinnooy Kan 1985]

Two factors in the solution cost:

- traveling salesman tour cost **TSP**
- radial cost **rad** $:= \frac{2}{k} \cdot \sum_v \delta(v)$,
  where $\delta(v)$ is the v-to-depot distance



Properties on the optimal cost **OPT**:

- **OPT** $\geq$ **TSP**
- **OPT** $\geq$ **rad**

Implication: **2-approximation**

Analysis: Our Approach

Generalization of radial cost and TSP cost

For a real value R:

- **TSP**(R) := TSP cost on $\{v : \delta(v) \geq R\}$
- **rad**(R) := $\frac{2}{k} \sum_v \min\{\delta(v), R\}$

Structure Theorem

$$\mathbf{OPT} \geq \mathbf{TSP}(R) + \mathbf{rad}(R).$$

Implications:

- $R = 0 \implies$ **OPT** $\geq$ **TSP**
- $R = \infty \implies$ **OPT** $\geq$ **rad**
- R well chosen:

Nie and Z. 2024

1.55-approximation

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Thank you!